Kuwait University
Mathematics & Computer Science Dept.

Math 102 (Calculus II)
Second Examination

December 14, 2006
Duration: 90 minutes

Calculators, Mobile telephones and Pagers ARE NOT ALLOWED.

Answer all of the following questions.

1. Evaluate the following integrals. (each is worth 3 points)

(a) $\displaystyle\int x^5 e^{-x^3}\, dx.$

(b) $\displaystyle\int (\sec x)\,(\sin 2x)^3\, dx.$

(c) $\displaystyle\int \frac{1}{x^2\,\left(1+x^2\right)^{\frac{5}{2}}}\, dx.$

(d) $\displaystyle\int \frac{x}{\sqrt{x^2+2x-3}}\, dx.$

(e) $\displaystyle\int \frac{2x^2-10x+15}{x^3-4x^2+5x}\, dx.$

(f) $\displaystyle\int \frac{dx}{\tan x+\sin x}.$

(g) $\displaystyle\int \frac{\sqrt{x}-\sqrt[6]{x}}{\sqrt{x}\,(x+4\sqrt[3]{x})}\, dx.$

2. Determine whether the improper integral $\displaystyle\int_0^1 \frac{1}{x+\sqrt{x}}\, dx$ is convergent or divergent and find its value if convergent. (4 points)

Solution Key

1. (a) $\int x^5 e^{-x^3} dx = -\frac{1}{3}\int x^3(-3x^2e^{-x^3})\,dx = -\frac{1}{3}\int x^3(e^{-x^3})'dx = -\frac{1}{3}x^3e^{-x^3} + \int x^2e^{-x^3}dx$

$= -\frac{1}{3}x^3e^{-x^3} - \frac{1}{3}e^{-x^3} + C \qquad \left\{\text{Nomenclature} : u = x^3,\ dv = x^2e^{-x^3}dx\right\}$

(b) $\int (\sec x)\,(\sin 2x)^3\,dx = 8\int \cos^2 x \sin^3 x dx = 8\int \left(\cos^2 x\right)\left(1-\cos^2 x\right)\sin x dx$

$= -8\left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right) + C$

(c) $\int \frac{du}{x^2\,(1+x^2)^{\frac{5}{2}}} \overset{x=\tan\theta}{=} \int \frac{\sec^2\theta d\theta}{(\tan^2\theta)\sec^5\theta} = \int \frac{\left(\cos^5\theta\right)d\theta}{\sin^2\theta} = \int \frac{\left(1-\sin^2\theta\right)^2}{\sin^2\theta}\cos\theta\ d\theta \overset{u=\sin\theta}{=} \int \frac{(1-u^2)^2du}{u^2}$

$= \int (\frac{1}{u^2} - 2 + u^2)\ du = -\frac{1}{u} - 2u + \frac{1}{3}u^3 + C = -\frac{\sqrt{1+x^2}}{x} - \frac{2x}{\sqrt{1+x^2}} + \frac{1}{3}\frac{x^3}{(1+x^2)\sqrt{1+x^2}} + C$

(d) $\int \frac{x}{\sqrt{(x+1)^2-4}}dx \overset{x+1=2\sec\theta}{=} \int \frac{2\sec\theta-1}{2\tan\theta}2\sec\theta\tan\theta\,d\theta = \int (2\sec^2\theta - \sec\theta)d\theta$

$= 2\tan\theta - \ln|\sec\theta+\tan\theta| + C = \sqrt{x^2+2x-3} + \ln\left|\frac{x+1}{2} + \frac{\sqrt{x^2+2x-3}}{2}\right| + C$

(e) $x^3 - 4x^2 + 5x = x\,(x^2-4x+5)$, and x^2-4x+5 is irreducible.

$\frac{2x^2-10x+15}{x(x^2-4x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2-4x+5} \Rightarrow (A+B)x^2 + (-4A+C)x + 5A = 2x^2 - 10x + 15$

$A = 3,\ B = -1,\ C = 2$. So $\frac{2x^2-10x+15}{x(x^2-4x+5)} = \frac{3}{x} - \frac{x-2}{x^2-4x+5}$.

$\int (\frac{3}{x} - \frac{x-2}{x^2-4x+5})dx = 3\ln|x| - \frac{1}{2}\int \frac{2(x-2)}{x^2-4x+5}dx = 3\ln|x| - \frac{1}{2}\ln\left(x^2-4x+5\right) + C$

(f) $\int \frac{dx}{\tan x+\sin x} \overset{u=\tan\frac{x}{2}}{=} \int \frac{1}{\frac{2u}{1+u^2}\frac{1+u^2}{1-u^2} + \frac{2u}{1+u^2}}\frac{2}{1+u^2}du = \frac{1}{2}\int \frac{1-u^2}{u}du = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + C$

(g) $\int \frac{\sqrt{x}-\sqrt[6]{x}}{\sqrt{x}\left(x+4\sqrt[3]{x}\right)}dx \overset{u^6=x}{=} \int \frac{u^3-u}{u^3(u^6+4u^2)}6u^5du = 6\int \frac{u^3-u}{u^4+4}du = \frac{3}{2}\ln(u^4+4) - \frac{3}{2}\tan^{-1}\frac{u^2}{2} + C$

2. $\frac{1}{x+\sqrt{x}}$ is continuous on $(0,1]$, and $\lim_{x\to 0^+}\frac{1}{x+\sqrt{x}} = \infty$, so the integral is improper at 0.

$\int_t^1 \frac{1}{x+\sqrt{x}}dx = \int_t^1 \frac{1}{1+\sqrt{x}}\frac{1}{\sqrt{x}}dx = 2\,\ln(1+\sqrt{x})\big|_t^1 = 2[\ln 2 - \ln(1+\sqrt{t})]$

$\lim_{t\to 0^+}\int_t^1 \frac{1}{x+\sqrt{x}}dx = 2\ln 2$. The integral converges, and $\int_0^1 \frac{1}{x+\sqrt{x}}dx = 2\ln 2$.